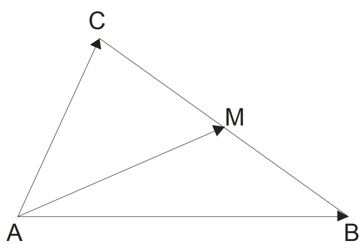
Vectors in the plane –part 2

Example one

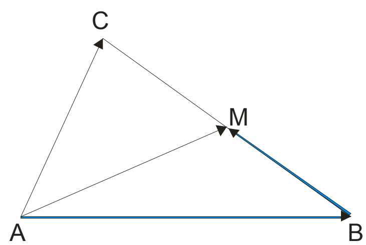
Point M is center of BC in triangle ABC. Prove that: $\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AM}$

Solution:

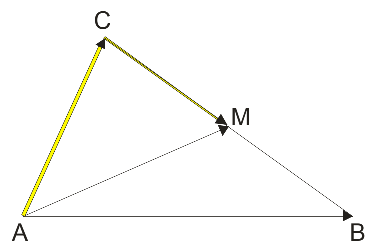
First draw a picture and set the problem...



picture 1.



picture 2.



picture 3.

At 1st picture mark vectors which are given in the task.

What is the idea?

In this type of tasks vector in the middle express at both sides!

picture 2. AM is a vector expressed (blue path): $\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{BM}$.

picture 3. AM is a vector expressed (yellow path): $\overrightarrow{AM}=\overrightarrow{AC}+\overrightarrow{CM}$

Next we write this two equality one under another , and gather them:

$\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{BM}$

$\underline{\overrightarrow{AM}=\overrightarrow{AC}+\overrightarrow{CM}}$

$2\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{BM}+\overrightarrow{AC}+\overrightarrow{CM}$ Arrange little

$2\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{AC}+\overrightarrow{CM}+\overrightarrow{BM}$ Look at the last two vectors in the picture ...opposite, $so\ \overrightarrow{CM}+\overrightarrow{BM}=0$

$2\overrightarrow{AM}=\overrightarrow{AB}+\overrightarrow{AC}$

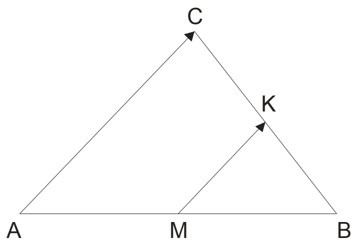
We have the required equality.

Example two

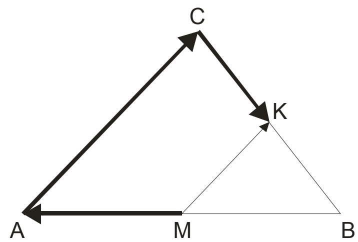
In triangle ABC, points M and K are the center of page AB and BC. Prove that: $\overrightarrow{AC} = 2\overrightarrow{MK}$

Solution:

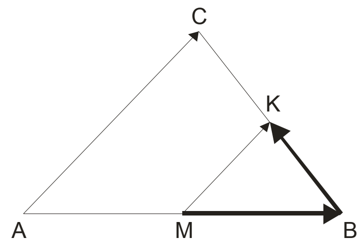
Draw:



picture 1.



picture 2.



picture 3.

Similarly as in the previous task, a vector in the middle MK, we express on both sides.
At picture 2. we go left: $\overrightarrow{KM} = \overrightarrow{MA} + \overrightarrow{AC} + \overrightarrow{CK}$

At picture 3. we go right: $\overrightarrow{KM} = \overrightarrow{MB} + \overrightarrow{BK}$

$$\overrightarrow{KM} = \overrightarrow{MA} + \overrightarrow{AC} + \overrightarrow{CK}$$
$$\underline{\overrightarrow{KM} = \overrightarrow{MB} + \overrightarrow{BK}}$$
$$2\overrightarrow{KM} = \overrightarrow{MA} + \overrightarrow{AC} + \overrightarrow{CK} + \overrightarrow{MB} + \overrightarrow{BK}$$

Now look at the picture and recognize the opposite vectors (the same intensity and opposite direction).

$$2\overrightarrow{KM} = \overrightarrow{AC} + \boxed{\overrightarrow{MA} + \overrightarrow{MB}} + \boxed{\overrightarrow{BK} + \overrightarrow{CK}}$$
in $\square$ are zero vectors, so:

$$2\overrightarrow{KM} = \overrightarrow{AC}$$

Example three

Trapeze ABCD is given . If M is the center of the pageAD, N is the center of the page BC, then $\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right)$ **Prove.**

Solution:

This is the proof of the fact that the median trapezoid is half the sum of base $m = \frac{a+b}{2}$

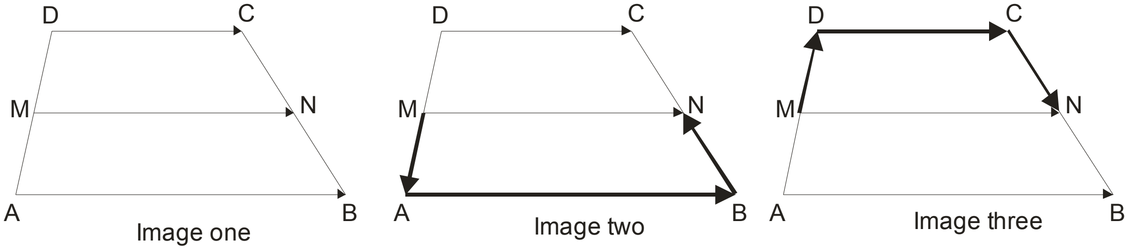


Image one Image two Image three

First, we express the vector MN down (image two) and up (image three), and gather …

$$\overrightarrow{MN} = \cancel{\overrightarrow{MA}} + \overrightarrow{AB} + \cancel{\overrightarrow{BN}}$$
$$\overrightarrow{MN} = \cancel{\overrightarrow{MD}} + \overrightarrow{DC} + \cancel{\overrightarrow{CN}}$$
$$2\overrightarrow{MN} = \overrightarrow{AB} + \overrightarrow{DC}$$

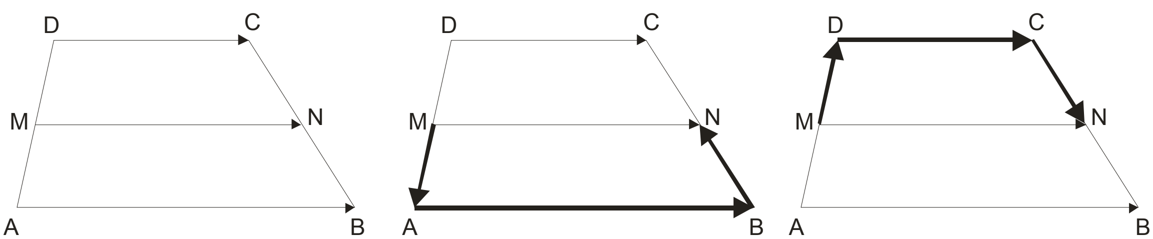
The entire equity devide with 2, and get $\overrightarrow{MN} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{CD}\right)$.

Example four

Let the M and N be centers of unparalleled pages BC and AD on trapezoid ABCD, E and F intersecting point of a long diagonal AC and BD. Then $\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{DC}\right)$

Solution:

In this task we wii use the **same trick**.



picture 1. picture 2. picture 3.

In picture 2. EF vector express through: $\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF}$.

In picture 3. EF vector express through: $\overrightarrow{EF} = \overrightarrow{EC} + \overrightarrow{CD} + \overrightarrow{DF}$

Write the two equal one below the other, gather them and ask the opposing vectors ...

$$\overrightarrow{EF} = \cancel{\overrightarrow{EA}} + \overrightarrow{AB} + \cancel{\overrightarrow{BF}}$$
$$\overrightarrow{EF} = \cancel{\overrightarrow{EC}} + \overrightarrow{CD} + \cancel{\overrightarrow{DF}}$$
$$2\overrightarrow{EF} = \overrightarrow{AB} + \overrightarrow{CD}$$

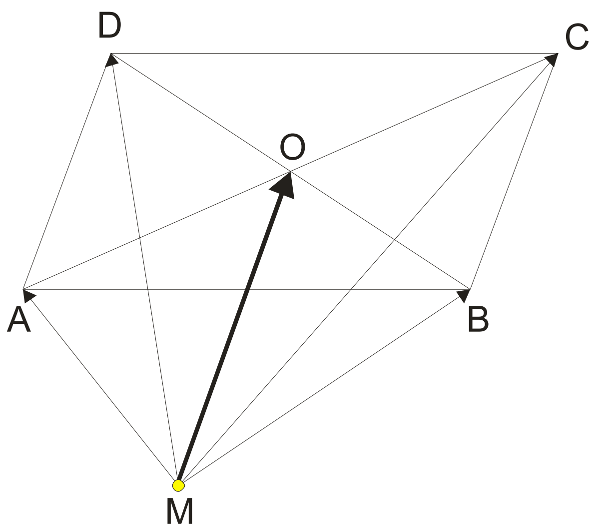
We know that is true: $\overrightarrow{CD} = -\overrightarrow{DC}$, throw this into the resulting equality and behold solutions: $2\overrightarrow{EF} = \overrightarrow{AB} - \overrightarrow{DC}$.

Of course, this all divide by 2 and get: $\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{DC}\right)$.

Example five

If M is an arbitrary point in the plane of the parallelogram ABCD, then $4\overrightarrow{MO} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD}$**. Prove.**

Solution:



MO vector will express in 4 ways and gather that :

$$\overrightarrow{MO} = \overrightarrow{MA} + \overrightarrow{AO}$$
$$\overrightarrow{MO} = \overrightarrow{MB} + \overrightarrow{BO}$$
$$\overrightarrow{MO} = \overrightarrow{MC} + \overrightarrow{CO}$$
$$\underline{\overrightarrow{MO} = \overrightarrow{MD} + \overrightarrow{DO}}$$
$$4\overrightarrow{MO} = \overrightarrow{MA} + \overrightarrow{AO} + \overrightarrow{MB} + \overrightarrow{BO} + \overrightarrow{MC} + \overrightarrow{CO} + \overrightarrow{MD} + \overrightarrow{DO}$$

$4\overrightarrow{MO} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} + \boxed{\overrightarrow{AO} + \overrightarrow{CO}} + \boxed{\overrightarrow{BO} + \overrightarrow{DO}}$ (see in the picture, these are opposite vectors)

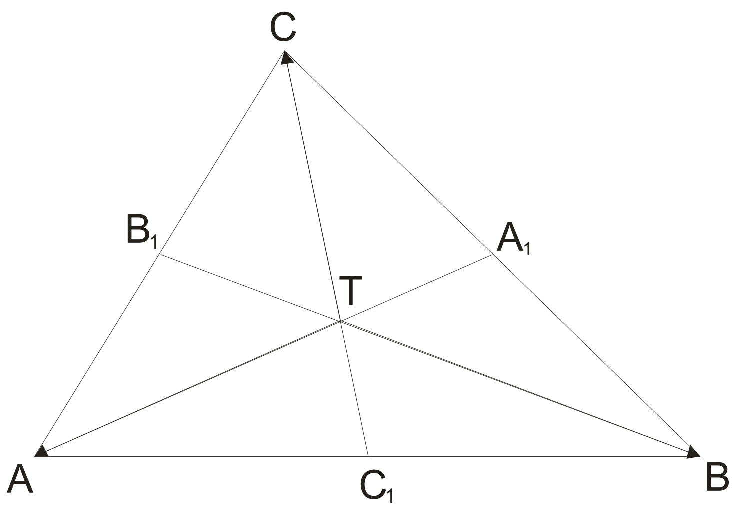
$$4\overrightarrow{MO} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD}$$

Example six

If T is the focus of the triangle ABC, then $\overrightarrow{TA}+\overrightarrow{TB}+\overrightarrow{TC}=0$. Prove.

Solution:

To draw a picture:



Start from $\overrightarrow{TA}+\overrightarrow{TB}+\overrightarrow{TC}=$ and prove that this sum is zero.

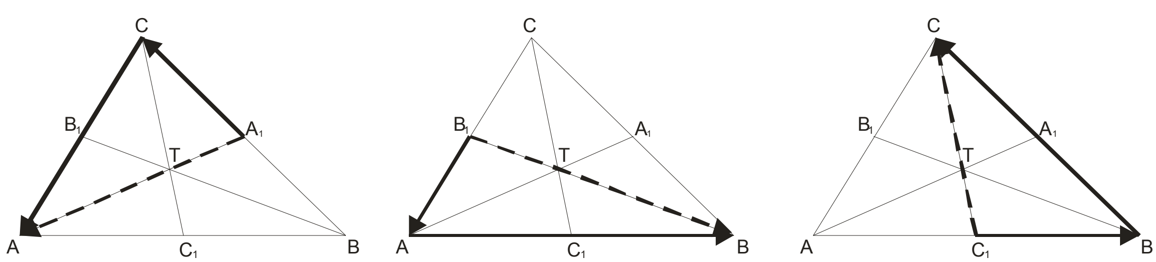
We know that the focus divided along mostly of relative 2:1, so:

$\overrightarrow{TA}=\frac{2}{3}\overrightarrow{A_1A}$

$\overrightarrow{TB}=\frac{2}{3}\overrightarrow{B_1B}$ Gather these three equality and obtain: $\overrightarrow{TA}+\overrightarrow{TB}+\overrightarrow{TC}=\frac{2}{3}(\overrightarrow{A_1A}+\overrightarrow{B_1B}+\overrightarrow{C_1C})$

$\overrightarrow{TC}=\frac{2}{3}\overrightarrow{C_1C}$

Next we will express each of these vectors (see in the picture, they are discontinuous drawing vectors):



$\overrightarrow{A_1A}=\overrightarrow{A_1C}+\overrightarrow{CA}$ $\overrightarrow{B_1B}=\overrightarrow{B_1A}+\overrightarrow{AB}$ $\overrightarrow{C_1C}=\overrightarrow{C_1B}+\overrightarrow{BC}$

Gather the three equality:

$$\overrightarrow{A_1A} = \overrightarrow{A_1C} + \overrightarrow{CA}$$
$$\overrightarrow{B_1B} = \overrightarrow{B_1A} + \overrightarrow{AB}$$
$$\overrightarrow{C_1C} = \overrightarrow{C_1B} + \overrightarrow{BC}$$
$$\overrightarrow{A_1A} + \overrightarrow{B_1B} + \overrightarrow{C_1C} = \overrightarrow{A_1C} + \overrightarrow{CA} + \overrightarrow{B_1A} + \overrightarrow{AB} + \overrightarrow{C_1B} + \overrightarrow{BC}$$

solve right side of equality:

$$\overrightarrow{A_1A} + \overrightarrow{B_1B} + \overrightarrow{C_1C} = \boxed{\overrightarrow{CA} + \overrightarrow{AB} + +\overrightarrow{BC}} + \overrightarrow{A_1C} + \overrightarrow{B_1A} + \overrightarrow{C_1B}$$

Rounded vectors have zero sum, because the last vector ends where the first begins...

Now the remaining sum $\overrightarrow{A_1C} + \overrightarrow{B_1A} + \overrightarrow{C_1B}$, and it is zero, because:

$$\overrightarrow{A_1C} = \frac{1}{2}\overrightarrow{BC}$$
$$\overrightarrow{B_1A} = \frac{1}{2}\overrightarrow{CA}$$
$$\overrightarrow{C_1B} = \frac{1}{2}\overrightarrow{AB}$$
$$\overrightarrow{A_1C} + \overrightarrow{B_1A} + \overrightarrow{C_1B} = \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}) = \frac{1}{2}\cdot 0 = 0$$

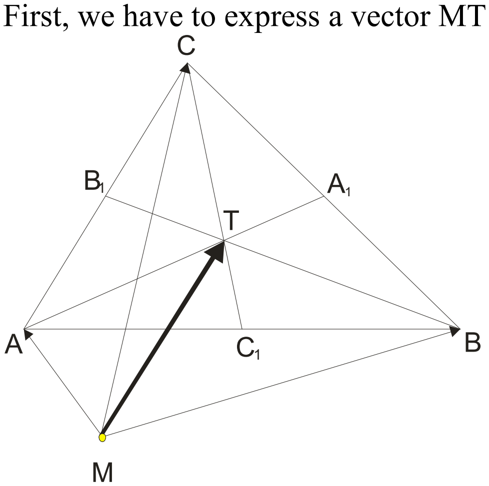
Well, this proof is finally finished.

Exsamlpe 7.

If M is an arbitrary point in the plane triangle ABC, then $\overrightarrow{MT} = \frac{1}{3}\left(\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}\right)$**, where T is the focus of the triangle. Prove.**

Solution:

First, we have to express a vector MT through the vector MA, MB and MC.



$\overrightarrow{MT} = \overrightarrow{MA} + \overrightarrow{AT}$

$\overrightarrow{MT} = \overrightarrow{MB} + \overrightarrow{BT}$ gather the three equality...

$\overrightarrow{MT} = \overrightarrow{MC} + \overrightarrow{CT}$

$$\begin{aligned}
\overrightarrow{MT} &= \overrightarrow{MA} + \overrightarrow{AT} \\
\overrightarrow{MT} &= \overrightarrow{MB} + \overrightarrow{BT} \\
\underline{\overrightarrow{MT}} &\underline{= \overrightarrow{MC} + \overrightarrow{CT}} \\
3\overrightarrow{MT} &= \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \boxed{\overrightarrow{AT} + \overrightarrow{BT} + \overrightarrow{CT}}
\end{aligned}$$

In the previous example we saw this framed provides a zero vector, and: $3\overrightarrow{MT} = \overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC}$, and that we should prove.